Fuzzy Geometries in M-theory

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hep-th/0504044 & hep/0605086

Some needless Quotes from Niels Bohr

- “Do not speak faster than you can think”
- “Einstein, please stop telling God what to do with his dice”
- “An expert is someone who has made lots of mistakes in a very narrow field”

Introduction

- Goals:

Learn some more about membranes and fivebranes

Obtain a description of membranes ending on fivebranes

Study an M-theory generalisation of noncommutative geometry

Branes ending on branes: k-D1, N-D3

D3 brane perspective

- ½ BPS solution of the world volume theory
- N=1, BIon solution to nonlinear theory, good approximation in large k limit
- N>1, Monopole solution to the U(N) gauge theory
- Spike geometry

D1 brane perspective

- ½ BPS solution of the world volume theory
- Require k>1, good approximation in large N limit.
- Fuzzy funnel geometry

D1 ending on a D3

- D3 brane perspective

$$D\phi = *F$$

Monopole equation

- D1 brane perspective

$$\frac{d\Phi^i}{d\sigma} = \pm\frac{i}{2}\epsilon_{ijk}[\Phi^j, \Phi^k]$$

Nahm Equation

Nahm equation

- Solution of the Nahm equation gives a fuzzy two sphere funnel:

$$\Phi^i = \hat{R}(\sigma)\alpha^i_N \,, \quad i = 1, 2, 3$$

Where

$$[\alpha^i_N, \alpha^j_N] = 2i\epsilon_{ijk}\alpha^k_N$$

and

$$\hat{R}(\sigma) = \pm\frac{1}{2(\sigma - \sigma_\infty)}$$

Fuzzy Funnel

- The radius of the two sphere is given by

$$R(\sigma)^2 = \frac{(2\pi l_s)^2}{N} \sum_{i=1}^{3} Tr[\Phi^i(\sigma)^2]$$

With

$$\sum_{i=1}^{3} (\alpha_N^i)^2 = (N^2 - 1) 1_{N \times N}$$

Which implies

$$R(\sigma) = \frac{N\pi\ell_s}{\sigma - \sigma_\infty} \sqrt{1 - \frac{1}{N^2}}$$

BIon Spike

- The BIon solution:

$$\phi(r) = \frac{\pi l_s N}{r}$$

- Agreement of the profile in the large N limit between BIon description and fuzzy funnel.
- Also, agreement between spike energy per unit length; Chern Simons coupling; and fluctuations.
- The Nahm Transform takes you between D1 and D3 brane descriptions of the system.

More on fuzzy spheres

- A fuzzy sphere is simply a sphere with a minimal distance scale
- The noncommutativity scale goes like 1/N so that one recovers ordinary geometry in the large N limit.
- Essentially one is representing the algebra of the sphere with finite N representations; the large N limit reproduces spherical harmonics.

More fuzzy spheres

- Consider harmonics on a 2-sphere with cutoff, E.
- Number of modes: $\sum_l^k (2l+1) = (k+1)^2$
- Where k is given by: $\frac{k(k+1)}{R^2} = E^2$
- If the radius R is given by: $R^2 = N^2 - 1$
- Then the number of modes in the large N limit scale as:

$$N^2$$

M2 branes ending on M5 branes

- D1 ending on D3 branes
- BIon Spike
- Nahm Equation
- Fuzzy Funnel with a two sphere blowing up into the D3

- M2 branes ending on M5 branes
- Self-dual string
- Basu-Harvey Equation
- Fuzzy Funnel with a three sphere blowing up in to M5

Self-dual string

Solution to the ½ BPS equation, $H = *d\phi$

BIon like spike gives the membrane

$$\phi(r) = \frac{cN}{r^2}$$

Basu-Harvey equation

$$\frac{dX^i}{ds} + \frac{M_{11}^3}{8\pi\sqrt{2N}} \frac{1}{4!} \epsilon_{ijkl} [G_5, X^j, X^k, X^l] = 0$$

Where

$$[X^1, X^2, X^3, X^4] = \sum_{perms\ \sigma} \mathrm{sign}(\sigma) [X^{\sigma(1)}, X^{\sigma(2)}, X^{\sigma(3)}, X^{\sigma(4)}]$$

And G_5 is a certain constant matrix

G_5

- G_5=generalisation of Γ^5

for higher dimensional representations

- It's the difference between the P+ projector and the P- projector acting on the symmetrised tensor product repesentation.
- For n=1, 4 dim rep. it is just Γ^5
- More generally, there will be representations given by $SU(2)_L$,$SU(2)_R$ labels:

(1/4(n+1),1/4(n-1))+(1/4(n-1),1/4(n+1))

- Dim of rep is N=1/2(n+1)(n+3)

Fuzzy funnel Solution

Solution:

$$X^i(s) = \frac{i\sqrt{2\pi}}{M_{11}^{\frac{3}{2}}}\frac{1}{\sqrt{s}}G^i$$

Where G_i obeys the equation of a fuzzy 3-sphere

$$G^i + \frac{1}{2(n+2)}\epsilon_{ijkl}G_5 G^j G^k G^l = 0$$

Properties of the solution

- The physical radius is given by

$$R = \sqrt{\left| \frac{Tr \sum (X^i)^2}{Tr 1} \right|}$$

Which yields

$$s \sim \frac{N}{R^2}$$

Agreeing with the self-dual string solution

From a Hamiltonian

- Consider the energy functional

$$E = \frac{T_2}{2}\int d^2\sigma Tr\left(X^{i'}X^{i'} - \frac{1}{3!}[X^j,X^k,X^l][X^j,X^k,X^l]\right)$$

- Bogmolnyi type construction yields

$$E = \frac{T_2}{2}\int d^2\sigma \left\{Tr\left(X^{i'} + g_{ijkl}\frac{1}{4!}[H^*,X^j,X^k,X^l]\right)^2 + T\right\}$$

$$T = -T_2\int d^2\sigma Tr\left(g_{ijkl}X^{i'}\frac{1}{4!}[H^*,X^j,X^k,X^l]\right)$$

From a Hamiltonian

- For more than 4 active scalars also require:

$$\frac{1}{3!} g_{ijkl} g_{ipqr} Tr\left([X^j, X^k, X^l][X^p, X^q, X^r]\right) = Tr\left([X^i, X^j, X^k][X^i, X^j, X^k]\right)$$

- H must have the properties:

$$\{H^*, X^i\} = 0 \quad H^{*2} = 1$$

- For four scalars one recovers B-H equation and $H=G_5$

Properties of this solution

- Just as for the D1 D3 system the fluctuation spectra matches and the tension matches.
- There is no equivalent of the Nahm transform.
- The membrane theory it is derived from is not understood.

Questions???

- Can the B-H equation be used to describe more than the M2 ending on a single M5?
- How do the properties of fuzzy spheres relate to the properties of nonabelian membranes?
- What is the relation between the B-H equation and the Nahm equation?
- Supersymmetry???
- How many degrees of freedom are there on the membrane?

M-theory Calibrations

- Configurations with less supersymmetry that correspond to intersecting M5 and M2 branes
- Classified by the calibration that may be used to prove that they are minimal surfaces
- Goal: Have the M2 branes blow up into generic M-theory calibrations

M-theory Calibrations

Planar five-brane

$$
\begin{array}{lccccccc}
M5: & 1 & 2 & 3 & 4 & 5 & & \\
M2: & 1 & & & & & & \#
\end{array}
$$

$$g_{2345} = 1 \qquad \nu = 1/2$$

$$X^{2'} = -H^*[X^3, X^4, X^5] \quad , \qquad X^{3'} = H^*[X^4, X^5, X^2] \; ,$$
$$X^{4'} = -H^*[X^5, X^2, X^3] \quad , \qquad X^{5'} = H^*[X^2, X^3, X^4] \; .$$

M-theory Calibrations

Intersecting five branes

$$\begin{array}{lllllllll} M5: & 1 & 2 & 3 & 4 & 5 & & & \\ M5: & 1 & 2 & 3 & & & 6 & 7 & \\ M2: & 1 & & & & & & & \# \end{array}$$

$$g_{2345} = g_{2367} = 1 \qquad \nu = 1/4$$

$$\begin{aligned} X^{2\prime} = -H^*[X^3, X^4, X^5] - H^*[X^3, X^6, X^7] \quad &, \quad X^{3\prime} = H^*[X^4, X^5, X^2] + H^*[X^6, X^7, X^2] \,, \\ X^{4\prime} = -H^*[X^5, X^2, X^3] \quad &, \quad X^{5\prime} = H^*[X^2, X^3, X^4] \,, \\ X^{6\prime} = -H^*[X^7, X^2, X^3] \quad &, \quad X^{7\prime} = H^*[X^2, X^3, X^6] \,, \\ [X^2, X^4, X^6] = [X^2, X^5, X^7] \quad &, \quad [X^2, X^5, X^6] = -[X^2, X^4, X^7] \,, \\ [X^3, X^4, X^6] = [X^3, X^5, X^7] \quad &, \quad [X^3, X^5, X^6] = -[X^3, X^4, X^7], \\ [X^4, X^5, X^6] = [X^4, X^5, X^7] \quad &= \quad [X^4, X^6, X^7] = [X^5, X^6, X^7] = 0. \end{aligned}$$

M-theory Calibrations

Intersecting five branes

$M5:$	1	2	3	4	5				
$M5:$	1	2	3			6	7		
$M5:$	1	2	3					8	9
$M2:$	1							$\#$	

$$g_{2345} = g_{2367} = g_{2389} = 1 \qquad \nu = 1/8$$

M-theory Calibrations

$$
\begin{aligned}
X^{2\prime} &= -H^*[X^3,X^4,X^5] \quad - \quad H^*[X^3,X^6,X^7] - H^*[X^3,X^8,X^9],\\
X^{3\prime} &= H^*[X^4,X^5,X^2] \quad + \quad H^*[X^6,X^7,X^2] + H^*[X^8,X^9,X^2]\ ,\\
X^{4\prime} &= -H^*[X^5,X^2,X^3] \quad , \quad X^{5\prime} = H^*[X^2,X^3,X^4]\ ,\\
X^{6\prime} &= -H^*[X^7,X^2,X^3] \quad , \quad X^{7\prime} = H^*[X^2,X^3,X^6]\ ,\\
X^{8\prime} &= -H^*[X^9,X^2,X^3] \quad , \quad X^{9\prime} = H^*[X^2,X^3,X^8]\ ,\\
[X^2,X^4,X^6] &= [X^2,X^5,X^7] \quad , \quad [X^2,X^5,X^6] = -[X^2,X^4,X^7],\\
[X^2,X^4,X^8] &= [X^2,X^5,X^9] \quad , \quad [X^2,X^5,X^8] = -[X^2,X^4,X^9]\ ,\\
[X^2,X^6,X^8] &= [X^2,X^7,X^9] \quad , \quad [X^2,X^7,X^8] = -[X^2,X^6,X^9],\\
[X^3,X^4,X^6] &= [X^3,X^5,X^7] \quad , \quad [X^3,X^5,X^6] = -[X^3,X^4,X^7],\\
[X^3,X^4,X^8] &= [X^3,X^5,X^9] \quad , \quad [X^3,X^5,X^8] = -[X^3,X^4,X^9]\ ,\\
[X^3,X^6,X^8] &= [X^3,X^7,X^9] \quad , \quad [X^3,X^7,X^8] = -[X^3,X^6,X^9],\\
[X^4,X^5,X^6] + [X^6,X^8,X^9] &= 0 \quad , \quad [X^4,X^5,X^7] + [X^7,X^8,X^9] = 0\ ,\\
[X^4,X^5,X^8] + [X^6,X^7,X^8] &= 0 \quad , \quad [X^4,X^5,X^9] + [X^6,X^7,X^9] = 0\ ,\\
[X^4,X^6,X^7] + [X^4,X^8,X^9] &= 0 \quad , \quad [X^5,X^6,X^7] + [X^5,X^8,X^9] = 0\ ,\\
[X^4,X^6,X^8] &= [X^4,X^7,X^9] \quad + \quad [X^5,X^6,X^9] + [X^5,X^7,X^8]\ ,\\
[X^5,X^7,X^9] &= [X^5,X^6,X^8] \quad + \quad [X^4,X^7,X^8] + [X^4,X^6,X^9]\ .
\end{aligned}
$$

M-theory Calibrations

$$
\begin{array}{llllllllll}
M5: & 1 & 2 & 3 & 4 & 5 & & & \\
M5: & 1 & 2 & 3 & & & 6 & 7 & \\
M5: & 1 & & & 4 & 5 & 6 & 7 & \\
M2: & 1 & & & & & & & \#
\end{array}
$$

$$g_{2345} = g_{2367} = g_{4567} = 1 \qquad \nu = 1/8$$

M-theory Calibrations

$M5:$	1	2	3	4	5				
$M5:$	1	2		4		6		8	
$\bar{M}5:$	1	2	3			6	7		
$M5:$	1	2			5		7	8	
$M2:$	1								$\#$

$$g_{2345} = g_{2468} = -g_{2367} = g_{2578} = 1 \qquad \nu = 1/8$$

The solutions

- For example, two intersecting 5-branes

$$
\begin{aligned}
H^2 &= \mathrm{diag}\,(G^1, G^1) \\
H^3 &= \mathrm{diag}\,(G^2, G^2) \\
H^4 &= \mathrm{diag}\,(G^3, 0) \\
H^5 &= \mathrm{diag}\,(G^4, 0) \\
H^6 &= \mathrm{diag}\,(0, G^3) \\
H^7 &= \mathrm{diag}\,(0, G^4) \\
H^* &= \mathrm{diag}\,(G^5, G^5)
\end{aligned}
$$

This is a trivial superposition of the basic B-H solution.

There are more solutions to these equations corresponding to nonflat solutions.

Calibrations

- It is the calibration form g that goes into the generalised B-H equation.
- Fuzzy funnels can successfully described all sorts of five-brane configurations.
- Interesting to search for and understand the non-diagonal solutions.

Fuzzy Funnel description of membranes

- We have seen a somewhat ad hoc description of membranes ending on five-brane configurations. Is there any further indication that this approach may have more merit??
- Back to the basic M2 ending on an M5. The basic equation is that of a fuzzy 3-sphere.
- How many degrees of freedom are there on a fuzzy three sphere?

Fuzzy Three Sphere

- Again consider the number of modes of a three sphere with a fixed UV cut-off
- Number of modes scales as k^3 (large k limit)
- k is given by $k^2 = E^2 R^2$
- R is given by $R = \sqrt{N}$
- Number of Modes

$$N^{3/2}$$

Non-Abelian Membranes

- This recovers (surprisingly) the well known N dependence of the non-Abelian membrane theory (in the large N limit).
- The matrices in the action were originally just any NxN matrices but the solutions yielded a representation of the fuzzy three sphere.
- Other fuzzy three sphere properties:

1. The algebra of a fuzzy three sphere is nonassociative.
2. The associativity is recovered in the large N limit.

Relation to the Nahm Equation

- To relate the Basu-Harvey equation to the Nahm equation we do this by introducing a projection.
- Projection P should project out G_4 and then the remaining projected matrices obey the Nahm equation.
- Consider:

$$P = 1/2(1 + i\Gamma^4\Gamma^5)$$

Projecting to Nahm

- Properties:

$$P^2 = P \quad P\Gamma^4 P = P\Gamma^5 P = 0 \quad P\Gamma^a P = \Gamma^a \quad a = 1..3$$

$$P\Gamma^4\Gamma^5 P = incP$$

Apply to Basu-Harvey

Project the Basu-Harvey equation

$$P(\frac{dX^i}{ds} + \frac{M_{11}^3}{8\pi\sqrt{2N}}\frac{1}{4!}\epsilon_{ijkl}[G_5, X^j, X^k, X^l])P = 0$$

Case i=4, the equation vanishes

Case i=1,2,3 then one recovers the Nahm equation

Projected Basu-Harvey equation

- Provided:

$$X^4 = \frac{32\pi R_{11} G^4}{3c}$$

- Giving (in the large N limit)

$$\frac{dX^a}{d\sigma} + \frac{i}{2\alpha'}\epsilon_{abc}[X^b, X^c] = 0$$

Discussion

- Ad hoc attempts to generalise the Nahm equation have lead to interesting conjectures for the non-Abelian membrane theory.
- Successes include the incorporation of calibrations corresponding to various fivebrane intersections. The geometric profile, fluctuations and tensions match known results.
- The relation to the Nahm equation is through a projection (a bit different to the usual dimensional reduction.
- A key note of interest is the interpretation of the $N^{3/2}$ degrees of freedom of the membrane as coming from the fuzzy thee sphere.

Questions

- Can the membrane theory be made supersymmetric?
- Can we understand better the role of the nonassociativity of the 3-sphere fuzzy geometry?
- Are there futher tests that one can do?
- What about five-branes?