Three-Charge Black Holes on a Circle

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Based on hep-th/0606246 with:

Troels Harmark, Niels Obers and

Peter Rønne at NBI

Three-Charge Black Holes

Entropy of 5-dimensional extremal black hole with 3 charges

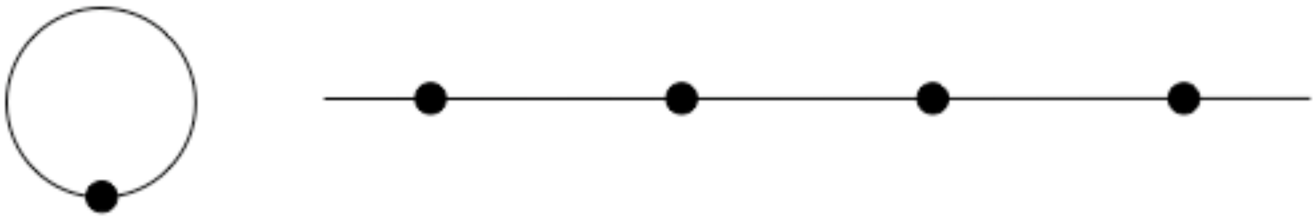
strong coupling: $S = \dfrac{A}{4G_N}$

weak coupling: $S = \log \Omega$

$$\left.\begin{array}{l} S = \dfrac{A}{4G_N} \\ S = \log \Omega \end{array}\right\} = 2\pi\sqrt{N_1 N_4 N_0}$$

Strominger, Vafa

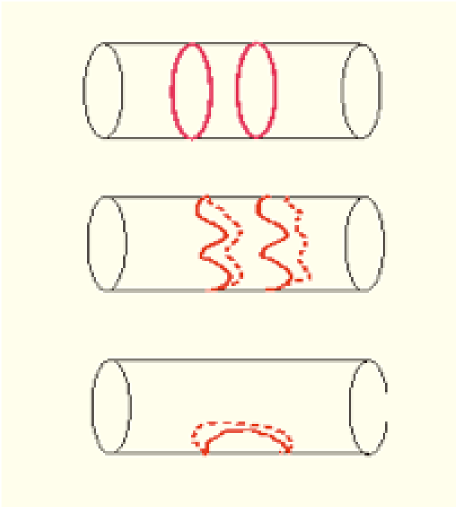
What happens to the entropy if we compactify one of the spatial dimensions on a circle?



BH on a circle — Covering space of the circle

Neutral Black Holes on a Circle

Spacetime asymptotes to $\mathcal{M}^4 \times S^1$

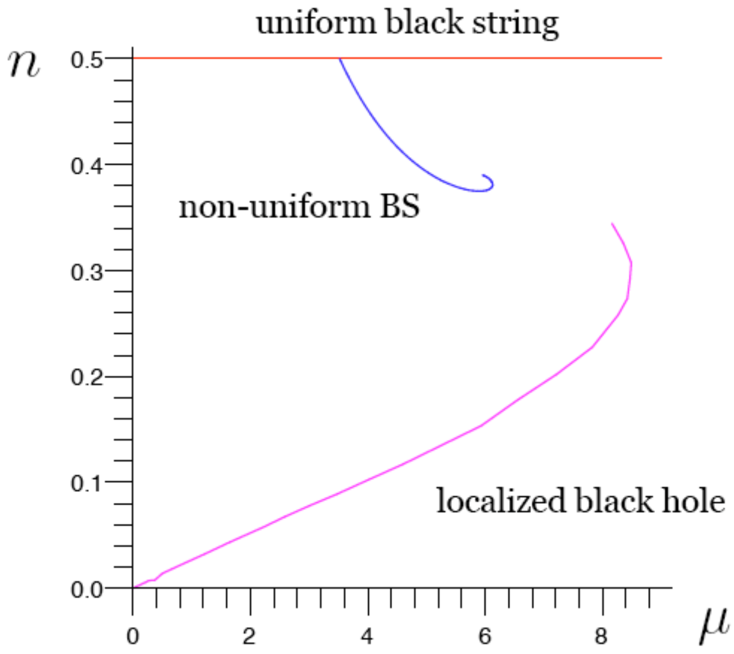


TWO physical quantities:

mass M

tension $\mathcal{T}$

(μ, n) phase diagram



$$\mu = \frac{16\pi G_N}{L^2} M$$

$$n = \frac{L\mathcal{T}}{M}$$

Harmark, Obers/Kol/
Wiseman, Kudoh/
Kleihaus, Kunz, Radu

$$ds^2 = -f dt^2 + \frac{L^2}{(2\pi^2)}\left[\frac{A}{f}dR^2 + \frac{A}{K^2}dv^2 + KR^2 d\Omega_2^2\right] \;, \quad f = 1 - \frac{R_0}{R}$$

Generating Charged Solutions

Generate solutions of 10-dim Type IIA SUGRA by "charging up" KKBHs.

Seeding solution:

$$ds_5^2 = -U dt^2 + \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b$$

- ✓ add flat dimensions
- ✓ boost in the new directions
- ✓ series of U-dualities

Three-charge solution:

$$ds_{10}^2 = H_1^{-\frac{3}{4}} H_4^{-\frac{3}{8}} H_0^{-\frac{7}{8}} \left(-U dt^2 + H_4 H_0 dx^2 + H_1 H_0 \sum_{i=1}^{4} (du^i)^2 + H_1 H_4 H_0 \frac{L^2}{(2\pi)^2} V_{ab} dx^a dx^b \right)$$

$$B = \coth\alpha_1 (H_1^{-1} - 1) dt \wedge dx,$$
$$A_{(5)} = \coth\alpha_4 (H_4^{-1} - 1) dt \wedge du^1 \wedge du^2 \wedge du^3 \wedge du^4,$$
$$A_{(1)} = \coth\alpha_0 (H_0^{-1} - 1) dt.$$

$$e^{2\phi} = H_1^{-1} H_4^{-\frac{1}{2}} H_0^{\frac{3}{2}}$$

$$H_a = 1 + (1 - U)\sinh^2\alpha_a$$

Compactify the new dimensions to get down to five dimensions again

Mapping of Physical Quantities

We can measure:

mass $\bar{M}$

three charges Q_a

tension $\bar{\mathcal{T}}_z$

Define dimensionless versions:

$$\bar{\mu} \equiv g\bar{M}$$

$$q_a \equiv gQ_a$$

$$\bar{n} \equiv \frac{L\bar{\mathcal{T}}_z}{\bar{M} - M^{\mathrm{el}}}$$

$$g \equiv \frac{16\pi G_{10}}{V_1 V_4 L^2}$$

Get a map from the neutral to the charged solutions

$$\bar{\mu} = \mu \left(1 + \frac{2-n}{3} (\sinh^2 \alpha_1 + \sinh^2 \alpha_4 + \sinh^2 \alpha_0) \right)$$

$$q_a = \mu \frac{2-n}{3} \sinh \alpha_a \cosh \alpha_a \qquad \bar{n} = n$$

Given the three charges we can write

$$\bar{\mu} = \sum_a q_a + \frac{1}{2}\mu n + \frac{(2-n)\mu}{6} \sum_a \frac{b_a}{1 + \sqrt{1 + b_a^2}} \quad \text{with} \quad b_a \equiv \frac{2-n}{6} \frac{\mu}{q_a}$$

The Near-Extremal Limit

- ✓ send charges to infinity
- ✓ size of circle to zero
- ✓ temperature finite

Size of the circle has same scale as energy above extremality

$$L \to 0, \quad \alpha_a \to \infty, \quad \ell_a \equiv L^{\gamma_a}\sqrt{q_a} = \text{fixed}, \quad g \equiv \frac{16\pi G_{10}}{V_1 V_4 L^2} = \text{fixed}$$

Must rescale all fields with appropriate power of $L/2\pi$

$$ds^2 = \hat{H}_1^{-\frac{3}{4}}\hat{H}_4^{-\frac{3}{8}}\hat{H}_0^{-\frac{7}{8}}\left(-U\,dt^2 + \hat{H}_4\hat{H}_0 dx^2 + \hat{H}_1\hat{H}_0\sum_{i=1}^{4}(du^i)^2 + \hat{H}_1\hat{H}_4\hat{H}_0 V_{ab}dx^a dx^b\right)$$

$$\hat{H}_a = \begin{cases} \hat{h}_a \frac{1-U}{\hat{c}_t} & \text{for } \gamma_a > 0 \\ H_a & \text{for } \gamma_a = 0 \end{cases}, \qquad \hat{h}_a \equiv \frac{(2\pi)^{1-2\gamma_a}\ell_a^2}{\Omega_2} \qquad e^{2\phi} = \hat{h}_1^{-1}\hat{h}_4^{-\frac{1}{2}}\hat{h}_0^{\frac{3}{2}}$$

$$(A_a)_{t\ldots} = \begin{cases} \hat{H}_a^{-1} & \text{for } \gamma_a > 0, \\ \coth\alpha_a(H_a^{-1} - 1) & \text{for } \gamma_a = 0. \end{cases}$$

Mapping of Physical Quantities

The asymptotic physical quantities are

Energy above extremality: $E = \lim_{L\to 0}\left(\bar{M} - \sum_a Q_a\right)$ or $\epsilon = gE$

(Relative) tension: $\hat{\mathcal{T}}_z = \lim_{L\to 0}\frac{L}{2\pi}\bar{\mathcal{T}}_z$ or $r = \frac{2\pi\hat{\mathcal{T}}_z}{E}$

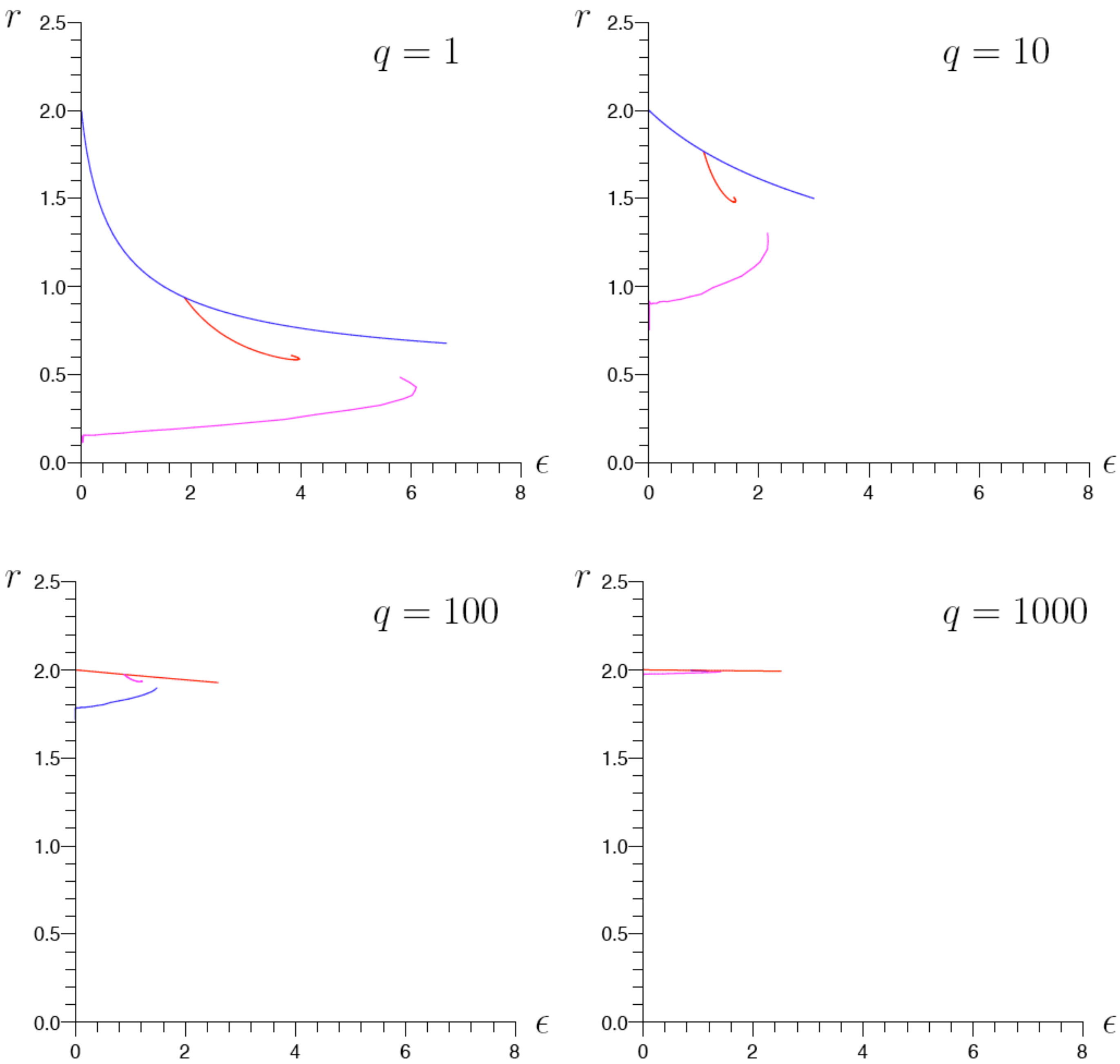
The map from seeding solution to near-extremal three charge

$$\epsilon = \frac{1}{2}\mu n, \qquad r = 2$$

The relative tension is a constant!

Can also map the rescaled thermodynamics

$$\hat{\mathfrak{t}} = \mathfrak{t}(\mathfrak{t}\mathfrak{s})^{3/2}, \quad \hat{\mathfrak{s}} = \mathfrak{s}(\mathfrak{t}\mathfrak{s})^{-3/2}$$



q = 1
q = 10
q = 100
q = 1000
r
ϵ
2.5
2.0
1.5
1.0
0.5
0.0
0
2
4
6
8

r = 2 and finite entropy

Turns out that r = const is a necessary condition for $S(\epsilon) \to S_0 \neq 0$

In general we can write $\epsilon = (a + bn)\mu$

1st law of thermodyn $\delta\epsilon = \hat{\mathfrak{t}}\delta\hat{\mathfrak{s}}$ and the Smarr relation $\hat{\mathfrak{t}}\hat{\mathfrak{s}} = \frac{2-n}{3}\mu$

give for small black holes $S \propto \epsilon^a \left(1 + \mathcal{O}(\epsilon)\right) \to 0$ unless $a = 0$

In turn: $r = \frac{\mu n}{(a+bn)\mu} = \frac{1}{b}$ if $a = 0$

In general we have $a = 1 - \frac{N_{\rm ch}(d-2)}{2(d-1)}$ and $b = \frac{N_{\rm ch}}{2(d-1)}$

Only find a = 0 for D = 5, N = 3 or D = 4, N = 4

Near-Extremal Thermodynamics

Uniform phase:

$$\hat{\mathfrak{s}}_{\rm u}(\epsilon) = \sqrt{2\epsilon}, \qquad \hat{\mathfrak{f}}_{\rm u}(\hat{\mathfrak{t}}) = -\frac{1}{2}\hat{\mathfrak{t}}^2.$$

Non-uniform phase:

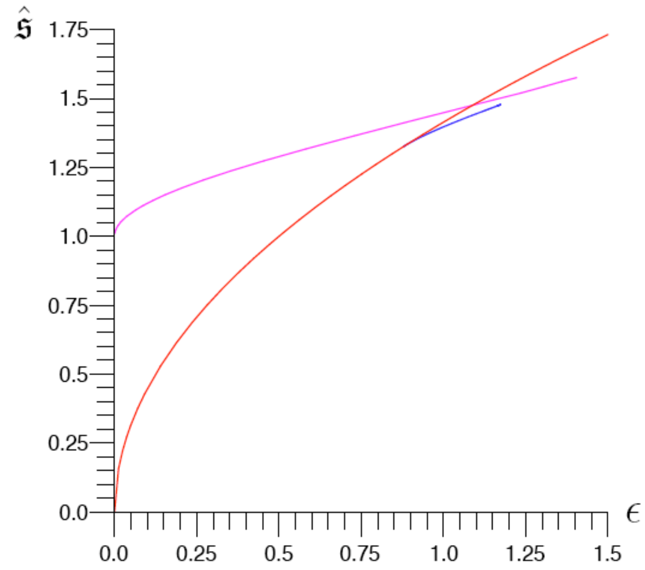
$$\hat{\mathfrak{s}}_{\rm nu}(\epsilon) = \hat{\mathfrak{s}}_{\rm c}\left(1 + \frac{\epsilon-\epsilon_{\rm c}}{2\epsilon_{\rm c}} - \left(\frac{1}{8} + \frac{2}{3}\hat{\gamma}\epsilon_{\rm c}\right)\frac{(\epsilon-\epsilon_{\rm c})^2}{\epsilon_{\rm c}^2}\right) + \mathcal{O}((\epsilon-\epsilon_{\rm c})^3)$$

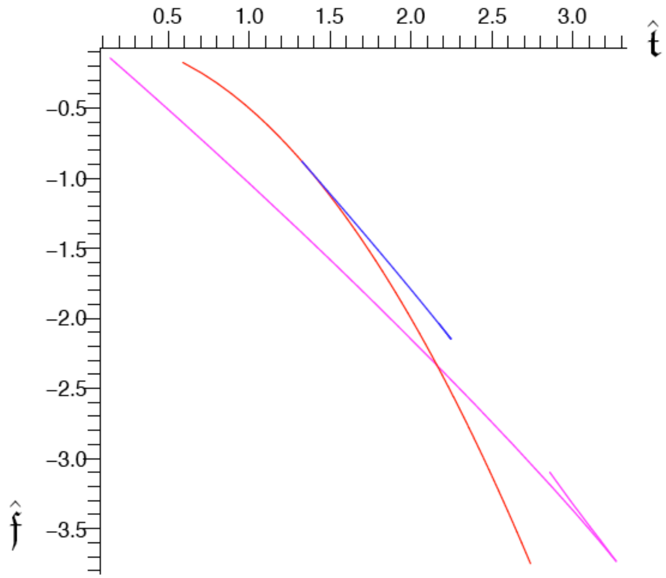
$$\hat{\mathfrak{f}}_{\rm nu} = -\epsilon_{\rm c} - \hat{\mathfrak{s}}_{\rm c}(\hat{\mathfrak{t}}-\hat{\mathfrak{t}}_{\rm c}) - \frac{c}{2\hat{\mathfrak{t}}_{\rm c}}(\hat{\mathfrak{t}}-\hat{\mathfrak{t}}_{\rm c})^2 + \mathcal{O}((\hat{\mathfrak{t}}-\hat{\mathfrak{t}}_{\rm c})^3)$$

Localized phase:

$$\hat{S}_{\rm loc} = \frac{\ell\hat{\mathfrak{s}}}{g} = 2\pi\sqrt{N_1N_4N_0}\left(1 + \sqrt{\frac{\epsilon}{8}} + \frac{\epsilon}{16} + \mathcal{O}(\epsilon^{3/2})\right)$$

$$\hat{\mathfrak{f}}_{\rm loc}(\hat{\mathfrak{t}}) = -\hat{\mathfrak{t}} - \frac{1}{32}\hat{\mathfrak{t}}^2 - \frac{1}{512}\hat{\mathfrak{t}}^3 + \mathcal{O}(\hat{\mathfrak{t}}^4)$$





Other Near-Extremal Limits

One finite charge:

will reproduce

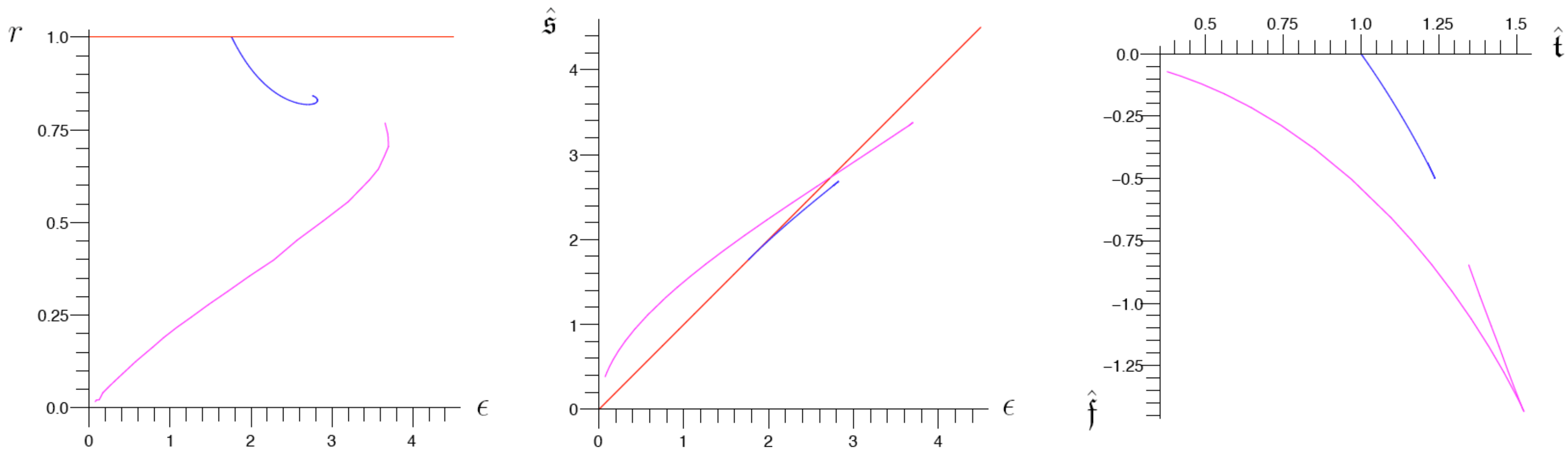
$$\epsilon = \rho_0^2 \sinh^2 \alpha_1 + \frac{1}{2}\rho_0^2 \left(1 + \frac{1}{16}\rho_0^2\right) + \mathcal{O}(\rho_0^6),$$

$$r = \frac{1}{8}\rho_0^2 - \frac{1}{128}\rho_0^4 + \mathcal{O}(\rho_0^6),$$

$$\hat{\mathfrak{s}} = \rho_0 \cosh \alpha_1 \left(1 + \frac{1}{16}\rho_0^2 + \frac{1}{512}\rho_0^4\right) + \mathcal{O}(\rho_0^7),$$

$$\hat{\mathfrak{t}} = \frac{\rho_0}{\cosh \alpha_1} \left(1 - \frac{1}{16}\rho_0^2 + \frac{1}{512}\rho_0^4\right) + \mathcal{O}(\rho_0^7).$$

Two near-extremal charges:



Microscopic Entropy

Without the circle we have

Extremal BH with N_1 strings, N_4 D4-branes, and N_0 D0-branes

$$S = 2\pi\sqrt{N_1 N_4 N_0}$$

Strominger, Vafa

Get non-extremality by adding $N_{\bar{0}}$ anti-D0-branes

$$S = 2\pi\sqrt{N_1 N_4}\left(\sqrt{N_0} + \sqrt{N_{\bar{0}}}\right)$$

Horowitz, Maldacena, Strominger

With the circle, the number of branes is shifted due to interactions

$$S = 2\pi\sqrt{N_1 N_4}\left(\sqrt{N_0'} + \sqrt{N_{\bar{0}}'}\right)$$

Costa, Perry

Need to find the effective number of D0-branes

Interaction Energy

The total mass of our non-extremal solution $\bar{M} = \bar{M}_1 + \bar{M}_4 + \bar{M}_0 + \tilde{E}$

$\bar{M}_a$ contribution from each type of object

$\tilde{E} = \frac{1}{2}\mathcal{T}L$ term proportional to the tension

Consider two extremal, one finite charge $\bar{M} = Q_1 + Q_4 + \bar{M}_0 + \tilde{E}$

Want to write this as $\bar{M} = Q_1 + Q_4 + \delta E + V_{\rm int}$

δE energy carried by D0 and anti-D0

$V_{\rm int} = -\int \mathcal{T} dL = -\frac{1}{2}\mathcal{T}L$ interaction energy

We find that $V_{\rm int} = -\tilde{E}$ so that: $\bar{M} = Q_1 + Q_4 + (\bar{M}_0 + 2\tilde{E}) + V_{\rm int}$

Can identify: $\delta E = \bar{M}_0 + 2\tilde{E}$

Finally, we require

$$\delta E \propto (N_0' + N_{\bar{0}}')$$
$$Q_0 \propto (N_0' - N_{\bar{0}}')$$

The effective number is shifted by $\tilde{E}$

Application to Localized BHs

If we apply this method to the localized black holes, we find

$$g\tau_0 N_0' = \frac{1}{4}\rho_0^2 \exp(2\alpha_0) + \frac{1}{32}\rho_0^4 + \mathcal{O}(\rho_0^6),$$
$$g\tau_0 N_{\bar{0}}' = \frac{1}{4}\rho_0^2 \exp(-2\alpha_0) + \frac{1}{32}\rho_0^4 + \mathcal{O}(\rho_0^6).$$

and therefore get

$$\sqrt{N_0'} + \sqrt{N_{\bar{0}}'} = \frac{\sqrt{N_0}}{\ell_0}\rho_0 \cosh\alpha_0 \left(1 + \frac{\rho_0^2}{16} + \mathcal{O}(\rho_0^4)\right)$$

The microstate entropy becomes

$$S = \frac{2\pi\sqrt{N_1 N_4 N_0}}{\ell_0}\rho_0 \cosh\alpha_0 \left(1 + \frac{\rho_0^2}{16} + \mathcal{O}(\rho_0^4)\right)$$
$$= \frac{\ell_1 \ell_4}{g}\rho_0 \cosh\alpha_0 \left(1 + \frac{\rho_0^2}{16} + \mathcal{O}(\rho_0^4)\right).$$

Agrees with the Bekenstein-Hawking formula!

Conclusion and Outlook

Gravitational phase transition -> Gauge theory phase transition ?

Add three charges to the bubble-black hole sequences:

Is there a new stable phase?

Study the Hagedorn thermodynamics of two-charge case

Analyze unstable Gregory-Laflamme modes